

FACTORIZATIONS RELATED TO THE RECIPROCAL PASCAL MATRIX

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ABSTRACT. The reciprocal Pascal matrix has entries $\binom{i+j}{j}^{-1}$. Explicit formulæ for its LU-decomposition, the LU-decomposition of its inverse, and some related matrices are obtained. For all results, q -analogues are also presented.

1. INTRODUCTION

Recently, there has been some interest in the *reciprocal Pascal matrix* M , defined by

$$M_{i,j} = \binom{i+j}{j}^{-1};$$

the indices start here for convenience with $0, 0$, and the matrix is either infinite or has N rows and columns, depending on the context.

Richardson [6] has provided the decomposition $S = GMG$, where the diagonal matrix G has entries $G_{i,i} = \binom{2i}{i}$, and S is the *super Catalan matrix* [2, 4] with entries

$$S_{i,j} = \frac{(2i)!(2j)!}{i!j!(i+j)!}.$$

We want to give an alternative decomposition of M , provided by the LU-decomposition. We will give explicit expressions for L and U , defined by $LU = M$, as well as for L^{-1} and U^{-1} .

Since there is also interest in M^{-1} , in particular in the integrality of its coefficients, we also provide the LU-decomposition $AB = M^{-1}$, and give expressions for A , B , A^{-1} and B^{-1} .

In the last section, we provide q -analogues of these results.

2. IDENTITIES

The LU-decomposition $M = LU$ is given by

$$L_{i,j} = \frac{i!i!(2j)!}{(i+j)!(i-j)!j!j!}$$

and

$$U_{i,j} = \frac{(-1)^i j!j!i!(i-1)!}{(j+i)!(j-i)!(2i-1)!} \quad \text{for } i \geq 1.$$

For $i = 0$, the formula is $U_{0,j} = 1$.

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The formula that needs to be proved is

$$\sum_{0 \leq k \leq \min\{i,j\}} L_{i,k} U_{k,j} = \binom{i+j}{j}^{-1},$$

which is equivalent to

$$1 + \frac{2i!i!j!j!}{(2i)!(2j)!} \sum_{1 \leq k \leq \min\{i,j\}} (-1)^k \binom{2i}{i+k} \binom{2j}{j+k} = \binom{i+j}{j}^{-1}.$$

The von Szily identity [7, 2, 3] is

$$\frac{(2i)!(2j)!}{i!j!(i+j)!} = \sum_{k \in \mathbb{Z}} (-1)^k \binom{2i}{i+k} \binom{2j}{j+k},$$

and an equivalent form is, by symmetry,

$$\frac{(2i)!(2j)!}{i!j!(i+j)!} = \binom{2i}{i} \binom{2j}{j} + 2 \sum_{k \geq 1} (-1)^k \binom{2i}{i+k} \binom{2j}{j+k}.$$

Thus, the identity to be proven is now

$$\binom{i+j}{j} + \frac{i!j!(i+j)!}{(2i)!(2j)!} \left[\frac{(2i)!(2j)!}{i!j!(i+j)!} - \binom{2i}{i} \binom{2j}{j} \right] = 1,$$

which is obviously correct.

The formula for L^{-1} is for $i \geq j \geq 0$:

$$L_{i,j}^{-1} = \frac{(-1)^{i-j} i! i! (i+j-1)!}{(2i-1)!(i-j)! j! j!}.$$

If necessary ($i = j = 0$), this must be interpreted as a limit.

To check this, we consider

$$\begin{aligned} & \sum_k \frac{i!i!(2k)!}{(i+k)!(i-k)!k!k!} \frac{(-1)^{k-j} k!k!(k+j-1)!}{(2k-1)!(k-j)!j!j!} \\ &= \frac{2i!i!(-1)^j}{j!j!} \sum_{j \leq k \leq i} \frac{k}{(i+k)!(i-k)!} \frac{(-1)^k (k+j-1)!}{(k-j)!}. \end{aligned}$$

The sum can be evaluated by computer algebra (or otherwise), and the result is indeed $\llbracket i = j \rrbracket$, as desired.

The formula for U^{-1} is for $j \geq i \geq 1$

$$U_{i,j}^{-1} = \frac{(-1)^i (j+i)!(2j)!}{(j-i)!j!(j+i)!(j-1)!i!i!}$$

and for $i = 0$:

$$U_{0,j}^{-1} = \frac{(2j)!}{j!j!}.$$

The fact that $\sum_k U_{i,k} U_{k,j}^{-1} = \llbracket i = j \rrbracket$ can also be done by computer algebra. Since there are a few cases to be distinguished, it is omitted here.

The LU-decomposition $AB = M^{-1}$ depends on the dimension N and is given by

$$A_{i,j} = \frac{(-1)^{i-j}(N-j-1)!j!(N+i-1)!}{i!(N-i-1)!(N+j-1)!(i-j)!},$$

$$B_{i,j} = \frac{(-1)^{j+N-1}(N+j-1)!}{j!(j-i)!(N-j-1)!i!}.$$

Since M^{-1} does not have “nice” entries, we rather provide formulæ for A^{-1} and B^{-1} and prove the identity $B^{-1}A^{-1} = M$ instead. The results are:

$$A_{i,j}^{-1} = \frac{(N-j-1)!j!(N+i-1)!}{i!(N-i-1)!(N+j-1)!(i-j)!},$$

$$B_{i,j}^{-1} = \frac{(-1)^{j+N-1}(N-1-i)!j!i!}{(j-i)!(N+i-1)!}.$$

First we prove that these are indeed the inverses. We consider

$$\begin{aligned} & \sum_k \frac{(-1)^{i-k}(N-k-1)!k!(N+i-1)!}{i!(N-i-1)!(N+k-1)!(i-k)!} \frac{(N-j-1)!j!(N+k-1)!}{k!(N-k-1)!(N+j-1)!(k-j)!} \\ &= (-1)^i \frac{(N+i-1)!(N-j-1)!j!}{(N-i-1)!(N+j-1)!i!} \sum_{j \leq k \leq i} \frac{(-1)^k}{(i-k)!(k-j)!} \\ &= \frac{(N+i-1)!(N-j-1)!j!}{(N-i-1)!(N+j-1)!i!(i-j)!} \sum_{j \leq k \leq i} (-1)^{i-k} \binom{i-j}{i-k} \\ &= \frac{(N+i-1)!(N-j-1)!j!}{(N-i-1)!(N+j-1)!i!(i-j)!} \llbracket i = j \rrbracket = \llbracket i = j \rrbracket, \end{aligned}$$

which proves $AA^{-1} = I$. Similarly

$$\begin{aligned} & \sum_k \frac{(-1)^{k+N-1}(N-1-i)!k!i!}{(k-i)!(N+i-1)!} \frac{(-1)^{j+N-1}(N+j-1)!}{j!(j-k)!(N-j-1)!k!} \\ &= (-1)^j \frac{(N-1-i)!i!(N+j-1)!}{(N+i-1)!j!(N-j-1)!} \sum_k \frac{(-1)^k}{(k-i)!(j-k)!} \\ &= \frac{(N-1-i)!i!(N+j-1)!}{(N+i-1)!j!(N-j-1)!(j-i)!} \sum_k (-1)^{j-k} \binom{j-i}{j-k} \\ &= \frac{(N-1-i)!i!(N+j-1)!}{(N+i-1)!j!(N-j-1)!(j-i)!} \llbracket i = j \rrbracket = \llbracket i = j \rrbracket, \end{aligned}$$

which proves $B^{-1}B = I$.

Now we compute an entry in $B^{-1}A^{-1}$:

$$\begin{aligned} & \sum_k \frac{(-1)^{k+N-1}(N-1-i)!k!i!}{(k-i)!(N+i-1)!} \frac{(N-j-1)!j!(N+k-1)!}{k!(N-k-1)!(N+j-1)!(k-j)!} \\ &= (-1)^{N-1} \frac{(N-1-i)!i!(N-j-1)!j!}{(N+i-1)!(N+j-1)!} \sum_k \frac{(-1)^k(N+k-1)!}{(k-i)!(N-k-1)!(k-j)!} \end{aligned}$$

$$\begin{aligned}
&= (-1)^{N-1} \frac{i!j!(N-j-1)!}{(N+i-1)!} \sum_k (-1)^k \binom{N-1-i}{N-1-k} \binom{N+k-1}{N-1+j} \\
&= \frac{i!j!(N-j-1)!}{(N+i-1)!} \sum_k \binom{i-1-k}{N-1-k} \binom{N+k-1}{N-1+j} \\
&= \frac{i!j!(N-j-1)!}{(N+i-1)!} \sum_k \binom{i-1-k}{i-N} \binom{N+k-1}{N-1+j} \\
&= \frac{i!j!(N-j-1)!}{(N+i-1)!} \binom{i-1+N}{i+j} \\
&= \frac{i!j!}{(i+j)!} = M_{i,j},
\end{aligned}$$

as claimed.

Now we use the form $M^{-1} = AB$ and write the (i, j) entry:

$$\begin{aligned}
&\sum_k \frac{(N-k-1)!k!(N+i-1)!}{i!(N-i-1)!(N+k-1)!(i-k)!} \frac{(-1)^{j+N-1}(N+j-1)!}{j!(j-k)!(N-j-1)!k!} \\
&= \frac{(N+i-1)!(N+j-1)!}{i!(N-i-1)!j!(N-j-1)!} \sum_k \frac{(N-k-1)!}{(N+k-1)!(i-k)!} \frac{(-1)^{j+N-1}}{(j-k)!} \\
&= \binom{N-1}{i} \binom{N+j-1}{j} \sum_{0 \leq k \leq \min\{i,j\}} (-1)^{j+N-1} \binom{N+i-1}{i-k} \binom{N-k-1}{j-k}.
\end{aligned}$$

From this representation, it is clear that this is an integer. This was a question which was addressed in the affirmative in [6].

3. q -ANALOGUES

In this section we present q -analogues. Define $(q)_n := (1-q)(1-q^2) \dots (1-q^n)$, and

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{(q)_n}{(q)_k (q)_{n-k}};$$

these definitions are standard, see [1]. Then we have the following results.

$$\begin{aligned}
L_{i,j} &= \frac{(q)_i (q)_i (q)_{2j}}{(q)_{i+j} (q)_{i-j} (q)_j (q)_j}, \\
U_{i,j} &= \frac{(-1)^i q^{i(3i-1)/2} (1+q^i) (q)_j (q)_j (q)_i (q)_i}{(q)_{i+j} (q)_{j-i} (q)_{2i}} \quad \text{for } i \geq 1, \quad U_{0,j} = 1, \\
L_{i,j}^{-1} &= \frac{q^{i(i-1)/2} (-1)^{i-j} (q)_i (q)_i (q)_{i+j-1}}{(q)_{2i-1} (q)_{i-j}} \quad \text{for } j < i, \quad L_{i,i}^{-1} = 1, \\
U_{i,j}^{-1} &= \frac{(-1)^i q^{-j^2-ji+i(i+1)/2} (q)_{j+i-1} (q)_{2j} (q)_i (q)_i}{(q)_{j-i} (q)_j (q)_{j-1}} \quad \text{for } j > i,
\end{aligned}$$

$$\begin{aligned}
 U_{i,i}^{-1} &= \frac{(-1)^i q^{i(3i+1)/2} (q)_{2i} (q)_{2i}}{(q)_i (q)_i (q)_i (q)_i (1+q^i)} \quad \text{for } i \geq 1, \quad U_{0,0}^{-1} = 1, \\
 A_{i,j} &= \frac{(-1)^{i-j} q^{(i+j+3)(i-j)/2 + N(j-i)} (q)_{N-j-1} (q)_j (q)_{N+i-1}}{(q)_{N-i-1} (q)_i (q)_{N+j-1} (q)_{i-j}}, \\
 B_{i,j} &= \frac{(-1)^{j+N-1} q^{i^2+j(j+3)/2 - Nj - N(N-1)/2} (q)_{N+j-1}}{(q)_j (q)_{j-i} (q)_{N-j-1} (q)_i}, \\
 A_{i,j}^{-1} &= \frac{q^{(i-j)(i-N+1)} (q)_{N-j-1} (q)_{N+i-1} (q)_j}{(q)_{N-i-1} (q)_{N+j-1} (q)_i (q)_{i-j}}, \\
 B_{i,j}^{-1} &= \frac{(-1)^{j+N-1} q^{j(j+1)/2 - (N-j-1)i - N(N-1)/2} (q)_{j-i} (q)_{N+i-1}}{(q)_{N-i-1} (q)_j (q)_i}.
 \end{aligned}$$

Note that for $q \rightarrow 1$, we get the previous formulæ. We do not discuss proofs here, since Zeilberger's algorithm (aka WZ-theory) [5] proves all these results (which were obtained by guessing), using a computer algebra system (such as, e. g., Maple).

REMARK. Richardson's decomposition $S = GMG$ still holds when all binomial coefficients are replaced by the corresponding Gaussian q -binomial coefficients.

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